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$W^{2,p}$-a priori estimates for the emergent Poincaré Problem

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Abstract We derive $W^{2,p}(\Omega)$-a priori estimates with *arbitrary* $p \in (1, \infty)$, for the solutions of a degenerate oblique derivative problem for linear uniformly elliptic operators with low regular coefficients. The boundary operator is given in terms of directional derivative with respect to a vector field $\boldsymbol{\ell}$ that is tangent to $\partial\Omega$ at the points of a non-empty set $\mathcal{E} \subset \partial\Omega$ and is of *emergent* type on $\partial\Omega$.

Keywords Uniformly elliptic operator · Poincaré problem · Emergent vector field · Strong solution · A priori estimates · L^p-Sobolev spaces

Mathematics Subject Classification (2000) Primary: 35J25 · 35R25; Secondary: 35R05 · 35B45 · 35H20 · 58J32

1 Introduction

The article deals with regularity in the Sobolev spaces $W^{2,p}(\Omega)$, $\forall\, p \in (1, \infty)$, of the strong solutions to the oblique derivative problem

$$\begin{aligned} &\mathcal{L}u := a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u = f(x) \qquad &&\text{a.e. } \Omega, \\ &\mathcal{B}u := \partial u/\partial\boldsymbol{\ell} = \varphi(x) &&\text{on } \partial\Omega, \end{aligned} \tag{$\mathcal{P}$}$$

where $\mathcal{L}$ is a uniformly elliptic operator with low regular coefficients and $\mathcal{B}$ is prescribed in terms of a directional derivative with respect to the unit vector field $\boldsymbol{\ell}(x) = (\ell^1(x), \dots, \ell^n(x))$ defined on $\partial\Omega$, $n \geq 3$. Precisely, we are interested in the Poincaré problem ($\mathcal{P}$) (cf. [16, 17, 20]), that is, a situation when $\boldsymbol{\ell}(x)$ becomes *tangential* to $\partial\Omega$ at the points of a non-empty subset $\mathcal{E}$ of $\partial\Omega$.

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From a mathematical point of view, $(\mathcal{P})$ is *not* an elliptic boundary value problem. In fact, it follows from the general PDEs theory that $(\mathcal{P})$ is a *regular (elliptic)* problem *if and only if* the Shapiro–Lopatinskij complementary condition is satisfied which means $\boldsymbol{\ell}$ must be transversal to $\partial\Omega$ when $n \geq 3$ and $|\boldsymbol{\ell}| \neq 0$ as $n = 2$. If $\boldsymbol{\ell}$ is *tangent* to $\partial\Omega$ then $(\mathcal{P})$ is a *degenerate* problem and new effects occur in contrast to the regular case. The qualitative properties of $(\mathcal{P})$ depend on the behaviour of $\boldsymbol{\ell}$ near the set of tangency $\mathcal{E}$ and especially on the way the normal component $\gamma\boldsymbol{\nu}$ of $\boldsymbol{\ell}$ changes or no its sign (with respect to the outward normal $\boldsymbol{\nu}$ to $\partial\Omega$) on the trajectories of $\boldsymbol{\ell}$ when these cross $\mathcal{E}$. The main results were obtained by Hörmander [5], Egorov and Kondrat'ev [1], Maz'ya [8], Maz'ya and Paneah [9], Melin and Sjöstrand [10], Paneah [15] and good surveys and details can be found in Popivanov and Palagachev [20] and Paneah [16]. The problem $(\mathcal{P})$ has been studied in the framework of Sobolev spaces $H^s (\equiv H^{s,2})$ assuming C^∞-smooth data and this naturally involved techniques from the pseudo-differential calculus.

The simplest case arises when $\gamma := \boldsymbol{\ell} \cdot \boldsymbol{\nu}$, even if zero on $\mathcal{E}$, conserves the sign on $\partial\Omega$ (Fig. 1). Then $\mathcal{E}$ and $\boldsymbol{\ell}$ are of *neutral* type (a terminology coming from the physical interpretation of $(\mathcal{P})$ in the theory of Brownian motion, cf. [20]) and $(\mathcal{P})$ is a problem of Fredholm type [1]. Assume now that γ changes the sign from "−" to "+" in positive direction along the $\boldsymbol{\ell}$-integral curves through the points of $\mathcal{E}$. Then $\boldsymbol{\ell}$ is of *emergent* type and $\mathcal{E}$ is called *attracting* manifold. The new effect occurring now is that the *kernel* of $(\mathcal{P})$ is *infinite-dimensional* [5] and to get a well-posed problem one has to modify $(\mathcal{P})$ by prescribing the values of u on $\mathcal{E}$ (cf. [1]). Finally, suppose the sign of γ changes from "+" to "−" along the $\boldsymbol{\ell}$-trajectories. Now $\boldsymbol{\ell}$ is of *submergent* type and $\mathcal{E}$ corresponds to a *repellent* manifold. The problem $(\mathcal{P})$ has *infinite-dimensional cokernel* [5] and Maz'ya and Paneah [9] were the first to propose a relevant modification of $(\mathcal{P})$ by violating the boundary condition at the points of $\mathcal{E}$. As result, a Fredholm problem arises, but the restriction $u|_{\partial\Omega}$ has a finite jump at $\mathcal{E}$. What is the common feature of the degenerate problems, independently of the type of $\boldsymbol{\ell}$, is that the solution "loses regularity" near the set of tangency from the data of $(\mathcal{P})$ in contrast to the non-degenerate case when each solution gains two derivatives from f and one derivative from φ. Roughly speaking, that loss of smoothness depends on the *order of contact* between $\boldsymbol{\ell}$ and $\partial\Omega$ and is given by the *subelliptic* estimates obtained for the solutions of degenerate problems (cf. [3–5,9]). Precisely, if $\boldsymbol{\ell}$ has a contact of order k with $\partial\Omega$ then the solution of $(\mathcal{P})$ gains $2 - k/(k+1)$ derivatives from f and $1 - k/(k+1)$ derivatives from φ.

For what concerns the geometric structure of $\mathcal{E}$, it was supposed initially to be a submanifold of $\partial\Omega$ of codimension one. Melin and Sjöstrand [10] and Paneah [15] were the first to study the Poincaré problem $(\mathcal{P})$ in a more general situation when $\mathcal{E}$ is a massive subset of $\partial\Omega$ with positive surface measure, allowing $\mathcal{E}$ to contain arcs of $\boldsymbol{\ell}$-trajectories of *finite* length. These results were extended to Hölder's spaces by Winzell [21,22] who studied $(\mathcal{P})$ assuming $C^{1,\alpha}$-smoothness of the coefficients of $\mathcal{L}$.

When dealing with non-linear Poincaré problems, however, we have to dispose of precise information on the linear problem $(\mathcal{P})$ with coefficients less regular than C^∞ (see [11,18–20]). Indeed, a priori estimates in $W^{2,p}$ for solutions to $(\mathcal{P})$ would imply easily pointwise

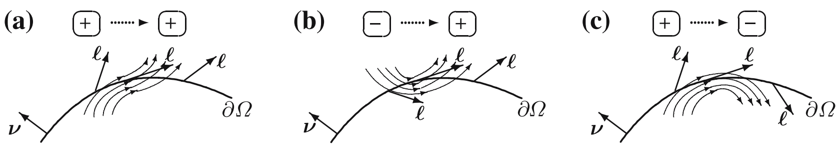

Fig. 1 Neutral (**a**), Emergent (**b**) and Submergent (**c**) vector field $\boldsymbol{\ell}$

estimates for u and Du for suitable values of $p > 1$ through the Sobolev imbeddings. This way, we are naturally led to consider ($\mathcal{P}$) in a *strong* sense, that is, to searching for solutions from $W^{2,p}$ which satisfy $\mathcal{L}u = f$ almost everywhere (a.e.) in Ω and $\mathcal{B}u = \varphi$ holds in the sense of trace on $\partial\Omega$.

In the papers [3,4] by Guan and Sawyer solvability and fine subelliptic estimates have been obtained for ($\mathcal{P}$) in $H^{s,p}$-spaces ($\equiv W^{s,p}$ for integer s!). However [3], treats operators with C^∞-coefficients and this determines the technique involved and the results obtained, while in [4] the coefficients are $C^{0,\alpha}$-smooth, but the field $\boldsymbol{\ell}$ is of finite type, that is, it has a *finite* order of contact with $\partial\Omega$.

The main goal of the article is to derive a priori estimates in Sobolev's classes $W^{2,p}(\Omega)$ with *any* $p \in (1,\infty)$ for the solutions to the Poincaré problem ($\mathcal{P}$), weakening both Winzell's assumptions on $C^{1,\alpha}$-regularity of the coefficients of $\mathcal{L}$ and these of Guan and Sawyer on the *finite type* of $\boldsymbol{\ell}$. We deal with the case of *emergent type* vector field $\boldsymbol{\ell}$ and, for the sake of simplicity, we suppose that $\mathcal{E}$ is a submanifold of $\partial\Omega$ of *codimension one*. As already mentioned, the *kernel of* ($\mathcal{P}$) *is infinite dimensional* and in order to get a well-posed problem we have to prescribe Dirichlet boundary condition on $\mathcal{E}$. Thus, we consider the modified Poincaré problem

$$\begin{aligned} &\mathcal{L}u = f(x) \quad \text{a.e. } \Omega, \\ &\mathcal{B}u = \varphi(x) \quad \text{on } \partial\Omega, \quad u = \mu(x) \quad \text{on } \mathcal{E} \end{aligned} \tag{$\mathcal{MP}$}$$

instead of ($\mathcal{P}$). Indeed, the loss of smoothness mentioned, imposes some more regularity of the data near the set $\mathcal{E}$. We assume that the coefficients of $\mathcal{L}$ are Lipschitz continuous near $\mathcal{E}$ while only continuity (and even discontinuity controlled in VMO) is allowed away from $\mathcal{E}$. Similarly, $\boldsymbol{\ell}$ is a Lipschitz vector field on $\partial\Omega$ with Lipschitz continuous first derivatives near $\mathcal{E}$, and *no restrictions* on the order of contact with $\partial\Omega$ are imposed.

Our main result is the a priori estimate from Theorem 1 for each $W^{2,p}(\Omega)$-solution to ($\mathcal{MP}$) with *arbitrary* $p \in (1,\infty)$. The background of our approach lies in the fact that $\partial u/\partial\boldsymbol{\ell}$ is a strong solution to a Dirichlet-type problem near $\mathcal{E}$ with right-hand side depending on the solution u itself. Precisely, let $\mathcal{N}$ be the manifold formed by the inward normals to $\partial\Omega$ starting from $\mathcal{E}$ and suppose $\boldsymbol{\ell}$ is appropriately extended in Ω. Thanks to the emergent type of $\boldsymbol{\ell}$, any point x near $\mathcal{N}$ could be reached from a unique $x' \in \mathcal{N}$ through an $\boldsymbol{\ell}$-trajectory and integration of $\partial u/\partial\boldsymbol{\ell}$ along it expresses $u(x)$ in terms of $u(x')$ and integral of $\partial u/\partial\boldsymbol{\ell}$ over the arc connecting x' and x. The supplementary condition $u|_{\mathcal{E}} = \mu$ provides for a $W^{2,p}(\mathcal{N})$-estimate for the restriction $u|_{\mathcal{N}}$ which solves a uniformly elliptic Dirichlet problem over the manifold $\mathcal{N}$. Since $\partial u/\partial\boldsymbol{\ell}$ is a local solution of a Dirichlet-type problem, the L^p-theory of such problems gives a bound for the $W^{2,p}$-norm of $\partial u/\partial\boldsymbol{\ell}$ in terms of the same norm of u. This way, a dynamical systems approach based on integration of these norms along the $\boldsymbol{\ell}$-trajectories through $\mathcal{N}$, leads to an estimate for the $W^{2,p}$-norm of u near $\mathcal{N}$, $\|u\|_{W^{2,p}}$, in terms of known quantities plus $C\|u\|_{W^{2,p}}$, where the multiplier C is small when the arclengths of the $\boldsymbol{\ell}$-trajectories joining x with x' are small. Indeed, that procedure gives an a priori bound for $\|u\|_{W^{2,p}}$ in a neighbourhood of $\mathcal{E}$. Away from $\mathcal{E}$, ($\mathcal{MP}$) is a regular oblique derivative problem and the $W^{2,p}(\Omega)$-a priori estimate follows from [7]. Another advantage of this approach is the *improving-of-integrability* property of the problem ($\mathcal{MP}$). Loosely speaking, it means that, even if ($\mathcal{MP}$) is a *degenerate* problem and therefore the solution loses derivatives from the data f and φ, it behaves as an *elliptic* problem for what concerns the degree p of integrability. That is, if $u \in W^{2,q}(\Omega)$ is a solution to ($\mathcal{MP}$) with $f \in L^p(\Omega)$ and $\partial f/\partial\boldsymbol{\ell} \in L^p$ near $\mathcal{E}$, $\varphi \in W^{1-1/p,p}(\partial\Omega)$ and $\varphi \in W^{2-1/p,p}$ near $\mathcal{E}$, $\mu \in W^{2-1/p,p}(\mathcal{E}_0)$ where $p \in [q,\infty)$, then $u \in W^{2,p}(\Omega)$.

Concluding this introduction, we refer the reader to the articles [12, 13, 14] , where various outgrowths of the $W^{2,p}(\Omega)$-a priori estimate and the *improving-of-integrability* property are derived for the Poincaré problem ($\mathcal{MP}$), such as maximum principle, uniqueness in $W^{2,p}(\Omega)$ for all $p > 1$, strong solvability when $c(x) \leq 0$ a.e. Ω, and it is proven that ($\mathcal{MP}$), even if a degenerate oblique derivative problem, is one of Fredholm type with index zero.

2 Improving of summability and $W^{2,p}$-a priori estimate

We are given a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 3$, with reasonably smooth boundary for which $\boldsymbol{\nu}(x) = \big(\nu^1(x), \ldots, \nu^n(x)\big)$ is the unit *outward* normal at the point $x \in \partial\Omega$. Let $\boldsymbol{\ell}(x) = \big(\ell^1(x), \ldots, \ell^n(x)\big)$ be a unit vector field defined on $\partial\Omega$ and decompose it into a sum of tangential and normal components, $\boldsymbol{\ell}(x) = \boldsymbol{\tau}(x) + \gamma(x)\boldsymbol{\nu}(x)$ at each $x \in \partial\Omega$. Here $\boldsymbol{\tau}(x), \boldsymbol{\tau}\colon \partial\Omega \to \mathbb{R}^n$, is the projection of $\boldsymbol{\ell}(x)$ on the tangent hyperplane to $\partial\Omega$ at the point $x \in \partial\Omega$ (see Fig. 2), while $\gamma\colon\partial\Omega \to \mathbb{R}$ stands for the Euclidean inner product $\gamma(x) := \boldsymbol{\ell}(x) \cdot \boldsymbol{\nu}(x)$. Indeed, the set of zeroes of the function $\gamma(x)$,

$$\mathcal{E} := \big\{x \in \partial\Omega\colon\ \gamma(x) = 0\big\}$$

is the subset of the boundary where the field $\boldsymbol{\ell}(x)$ becomes tangent to it.

Set further $\partial\Omega^{\pm}$ for the *relatively open* sets (see Fig. 2)

$$\partial\Omega^+ := \{x \in \partial\Omega\colon\ \gamma(x) > 0\}, \qquad \partial\Omega^- := \{x \in \partial\Omega\colon\ \gamma(x) < 0\}$$

so that $\mathcal{E}$ is the common boundary of $\partial\Omega^+$ and $\partial\Omega^-$, $\partial\Omega = \partial\Omega^+ \cup \partial\Omega^- \cup \mathcal{E}$ and $\operatorname{codim}_{\partial\Omega}\mathcal{E} = 1$. It is clear that $\partial\Omega^+$ is the set of all boundary points x where the field $\boldsymbol{\ell}(x)$ points *outwards* Ω, whereas it is pointed *inward* Ω on $\partial\Omega^-$. Regarding $\mathcal{E}$, we will suppose $\boldsymbol{\ell}$ is strictly transversal to it and directed from $\partial\Omega^-$ into $\partial\Omega^+$.

The standard summation convention on repeated indices is adopted throughout and $D_i := \partial/\partial x_i$, $D_{ij} := \partial^2/\partial x_i \partial x_j$. The class of functions with Lipschitz continuous kth order derivatives is denoted by $C^{k,1}$, $W^{k,p}$ stands for the Sobolev space of functions with L^p-summable weak derivatives up to order $k \in \mathbb{N}$ and normed by $\|\cdot\|_{W^{k,p}}$, while $W^{s,p}(\partial\Omega)$ with $s > 0$ non-integer, $p \in (1, +\infty)$, is the fractional-order Sobolev space. The Sarason class of functions

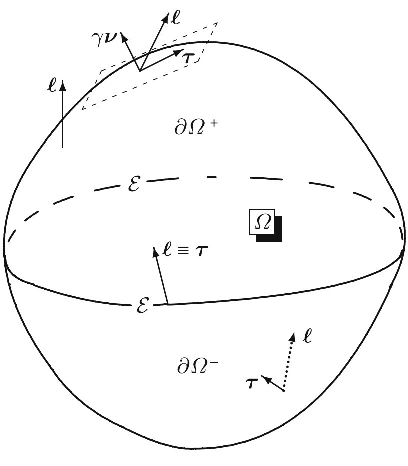

Fig. 2 The structure of the vector field $\boldsymbol{\ell}$

with vanishing mean oscillation is denoted by VMO(Ω). We use the standard parameterization $t \mapsto \boldsymbol{\psi}_{\boldsymbol{L}}(t,x)$ for the *trajectory* (*phase curve and maximal integral curve*) of a given vector field $\boldsymbol{L}$ passing through the point x, that is, $\partial_t \boldsymbol{\psi}_{\boldsymbol{L}}(t,x) = \boldsymbol{L} \circ \boldsymbol{\psi}_{\boldsymbol{L}}(t,x)$ and $\boldsymbol{\psi}_{\boldsymbol{L}}(0,x) = x$.

Fix hereafter $\Sigma \subset \overline{\Omega}$ to be a *closed* neighbourhood of $\mathcal{E}$ in $\overline{\Omega}$ and assume:

- *uniform ellipticity of the operator* $\mathcal{L}$*:* there exists a constant $\lambda > 0$ such that

$$\lambda^{-1}|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2, \quad a^{ij}(x) = a^{ji}(x) \quad \text{a.a. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n; \tag{1}$$

- *regularity of the data:*

$$\begin{aligned} &a^{ij} \in VMO(\Omega) \cap C^{0,1}(\Sigma) \equiv VMO(\Omega) \cap W^{1,\infty}(\Sigma),\\ &b^i,\ c \in L^\infty(\Omega) \cap C^{0,1}(\Sigma) \equiv L^\infty(\Omega) \cap W^{1,\infty}(\Sigma),\\ &\ell^i \in C^{0,1}(\partial\Omega) \cap C^{1,1}(\partial\Omega \cap \Sigma); \quad \partial\Omega \in C^{1,1}, \quad \partial\Omega \cap \Sigma \in C^{2,1}; \end{aligned} \tag{2}$$

- *emergent type of the vector field* $\boldsymbol{\ell}$*:*

$$\begin{aligned} &\mathcal{E} \text{ is a } C^{2,1}\text{-smooth submanifold of } \partial\Omega \text{ of codimension one and } \boldsymbol{\ell}(x)\\ &\text{is strictly transversal to } \mathcal{E},\ \text{pointing from } \partial\Omega^- \text{ into} \partial\Omega^+ \forall\, x \in \mathcal{E}. \end{aligned} \tag{3}$$

We will employ an extension of the field $\boldsymbol{\ell}$ near $\partial\Omega$ which preserves therein its regularity and geometric properties. For each $x \in \Omega$ and close enough to $\partial\Omega$ define $\Gamma := \{x \in \Omega\colon \operatorname{dist}(x,\partial\Omega) \le d_0\}$ with $d_0 > 0$ sufficiently small. Thus, to each $x \in \Gamma$ there corresponds a unique $y(x) \in \partial\Omega$ closest to x, $y(x) \in C^{0,1}(\Gamma)$ while $y(x) \in C^{1,1}(\Gamma \cap \Sigma)$ (cf. [2, Chap. 14]). We set

$$\boldsymbol{L}(x) := \boldsymbol{\ell}(y(x)), \quad \boldsymbol{\tau}(x) := \boldsymbol{\tau}(y(x)) \quad \forall\, x \in \Gamma, \qquad \mathcal{N} := \{x \in \Gamma:\ y(x) \in \mathcal{E}\}.$$

It is clear from (2) and (3) that $\boldsymbol{L},\ \boldsymbol{\tau} \in C^{0,1}(\Gamma) \cap C^{1,1}(\Gamma \cap \Sigma)$ and $\left|\boldsymbol{\tau}|_{\mathcal{N}}\right| = 1$ in view of $\boldsymbol{\tau}|_{\mathcal{N}} \equiv \boldsymbol{L}|_{\mathcal{N}} \equiv \boldsymbol{\ell}|_{\mathcal{E}}$. Moreover, $\mathcal{N}$ is a $C^{1,1}$-smooth manifold of dimension $(n-1)$ and the vector field $\boldsymbol{L}$ is *strictly transversal* to it.

In order to state our main results, we need to introduce special functional spaces which take into account the higher regularity near $\mathcal{E}$ of the data of ($\mathcal{MP}$). For any $p \in (1,\infty)$ define the Banach spaces

$$\mathcal{F}^p(\Omega,\Sigma) := \left\{f \in L^p(\Omega)\colon\ \partial f/\partial \boldsymbol{L} \in L^p(\Sigma)\right\}$$

equipped with the norm $\|f\|_{\mathcal{F}^p(\Omega,\Sigma)} := \|f\|_{L^p(\Omega)} + \|\partial f/\partial \boldsymbol{L}\|_{L^p(\Sigma)}$, and

$$\Phi^p(\partial\Omega,\Sigma) := \left\{\varphi \in W^{1-1/p,p}(\partial\Omega)\colon\ \varphi \in W^{2-1/p,p}(\partial\Omega \cap \Sigma)\right\}$$

normed by $\|\varphi\|_{\Phi^p(\partial\Omega,\Sigma)} := \|\varphi\|_{W^{1-1/p,p}(\partial\Omega)} + \|\varphi\|_{W^{2-1/p,p}(\partial\Omega\cap\Sigma)}$.

In the sequel the letter C will denote positive constants depending on the data of ($\mathcal{MP}$), that is, on n, p, λ, the respective norms of the coefficients of $\mathcal{L}$ and $\mathcal{B}$ in Ω and Σ, the regularity of $\partial\Omega$ and the lower bound for the angle between $\boldsymbol{\ell}$ and $\mathcal{E}$ [see (3)].

Our main result asserts that the couple $(\mathcal{L},\mathcal{B})$ *improves the integrability* of solutions to ($\mathcal{MP}$) for any $p \in (1,\infty)$ and provides for an a priori *estimate* in the L^p-Sobolev scales for any such solution.

Theorem 1 *Suppose* (1)–(3) *and let* $u \in W^{2,q}(\Omega)$ *be a strong solution to* ($\mathcal{MP}$) *with* $f \in \mathcal{F}^p(\Omega,\Sigma)$, $\varphi \in \Phi^p(\partial\Omega,\Sigma)$ *and* $\mu \in W^{2-1/p,p}(\mathcal{E})$ *where* $p \in [q,\infty)$.

Then $u \in W^{2,p}(\Omega)$ *and there is a constant C such that*

$$\|u\|_{W^{2,p}(\Omega)} \le C\left(\|u\|_{L^p(\Omega)} + \|f\|_{\mathcal{F}^p(\Omega,\Sigma)} + \|\varphi\|_{\Phi^p(\partial\Omega,\Sigma)} + \|\mu\|_{W^{2-1/p,p}(\mathcal{E})}\right). \tag{4}$$

Some remarks follow which regard the behaviour of $\partial u/\partial \boldsymbol{L}$ in a neighbourhood of the tangency set $\mathcal{E}$ and traces of functions on $(n-1)$-dimensional manifolds.

Remark 2 **(1)** The directional derivative $\partial u/\partial \boldsymbol{L}$ of any $W^{2,p}$-solution to ($\mathcal{MP}$) is a $W^{2,p}(\Sigma)$-function. In fact, $u \in W^{2,p}$ gives $\partial u/\partial \boldsymbol{L} \in W^{1,p}(\Sigma)$ and taking the difference quotients in $\boldsymbol{L}$-direction of the equation in ($\mathcal{MP}$), we get $\partial u/\partial \boldsymbol{L} \in W^{2,p}(\Sigma)$ in view of the regularity theory of uniformly elliptic equations (e.g. [2, Lemma 7.24,Chap. 8]). Moreover, $\partial u/\partial \boldsymbol{L}$ is a solution to the Dirichlet problem

$$\begin{aligned} \mathcal{L}(\partial u/\partial \boldsymbol{L}) &= \partial f/\partial \boldsymbol{L} + 2a^{ij}D_jL^kD_{ki}u + (a^{ij}D_{ij}L^k + b^iD_iL^k)D_ku \\ &\quad -(\partial a^{ij}/\partial \boldsymbol{L})D_{ij}u - (\partial b^i/\partial \boldsymbol{L})D_iu - (\partial c/\partial \boldsymbol{L})u \qquad \text{a.e. } \Sigma, \\ \partial u/\partial \boldsymbol{L} &= \varphi \qquad \text{on } \partial\Omega \cap \Sigma, \end{aligned} \tag{5}$$

where $\boldsymbol{L}(x) = \big(L^1(x), \dots, L^n(x)\big)$ is the $C^{0,1}(\Gamma) \cap C^{1,1}(\Gamma \cap \Sigma)$-extension of $\boldsymbol{\ell}$. Therefore, once having $u \in W^{2,p}(\Omega)$ and the estimate (4), the L^p-theory of the uniformly elliptic equations (cf. Chap. 9 in [2]) gives

$$\begin{aligned} \|\partial u/\partial \boldsymbol{L}\|_{W^{2,p}(\widetilde{\Sigma})} &\le C'\big(\|\partial u/\partial \boldsymbol{L}\|_{L^p(\Sigma)} + \|\partial f/\partial \boldsymbol{L}\|_{L^p(\Sigma)} + \|u\|_{W^{2,p}(\Sigma)} \\ &\qquad + \|\varphi\|_{W^{2-1/p,p}(\partial\Omega\cap\Sigma)}\big) \\ &\le C'\big(\|u\|_{L^p(\Omega)} + \|f\|_{\mathcal{F}^p(\Omega,\Sigma)} + \|\varphi\|_{\Phi^p(\partial\Omega,\Sigma)} + \|\mu\|_{W^{2-1/p,p}(\mathcal{E})}\big) \end{aligned} \tag{6}$$

for each *closed* neighbourhood $\widetilde{\Sigma}$ of $\mathcal{E}$ in $\overline{\Omega}$, $\widetilde{\Sigma} \subset \Sigma$, where the constant C' depends on dist $(\widetilde{\Sigma}, \Omega\setminus\Sigma)$ in addition. In other words, if a strong solution u to ($\mathcal{MP}$) belongs to $W^{2,p}(\Omega)$ then automatically $\partial u/\partial \boldsymbol{L} \in W^{2,p}(\Sigma)$ provided $f \in \mathcal{F}^p(\Omega, \Sigma)$ and $\varphi \in \Phi^p(\partial\Omega, \Sigma)$. Moreover, it will be evident from (5) and the proofs given below, that instead of the Lipschitz continuity of the coefficients of $\mathcal{L}$ in Σ as (2) asks, it suffices to have essentially bounded their $\boldsymbol{L}$-directional derivatives.

(2) Let $u \in L^p_{\text{loc}}(\mathbb{R}^n)$, $p > 1$, and let $\mathcal{N}$ be the $(n-1)$-dimensional manifold of the inward normals through the points of $\mathcal{E}$ constructed above, which can be represented locally as $\mathcal{N} = \{x \in \mathbb{R}^n\colon\ x_n = \Phi(x'),\ x' \in \mathcal{O}' \subset \mathbb{R}^{n-1}\}$ with $\Phi \in C^{1,1}(\mathcal{O}')$. Then the trace $u|_{\mathcal{N}}$ is not well-defined because $\mathcal{N}$ has zero n-dimensional Lebesgue measure. However, if $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ then $u|_{\mathcal{N}}$ exists and belongs to the fractional Sobolev space $W^{1-1/p,p}_{\text{loc}}(\mathcal{N})$. We are interested here on the intermediate situation when $u, \partial u/\partial x_n \in L^p_{\text{loc}}(\mathbb{R}^n)$. Then, redefining if necessary u on a set of zero measure, $u(x', x_n)$ is absolutely continuous function in x_n for a.a. x' and therefore $\partial u/\partial x_n(x', x_n)$ is a.e. classical derivative. This way, we define the trace $\widetilde{u}(x', \Phi(x')) := u|_{\mathcal{N}}$ by the formula

$$\widetilde{u}(x', \Phi(x')) = u(x', x_n) - \int_{\Phi(x')}^{x_n} \frac{\partial u}{\partial x_n}(x', s)\,\mathrm{d}s \qquad \text{a.a. } (x', \Phi(x')) \in \mathcal{N}. \tag{7}$$

It follows from Fubini's theorem that $\widetilde{u} \in L^p_{\text{loc}}(\mathcal{N})$. Moreover, having $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ with $\partial u/\partial x_n \in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ then $\widetilde{u} \in W^{2,p}_{\text{loc}}(\mathcal{H})$ and the trace operator $u \mapsto \widetilde{u}$ is compact one considered as mapping from $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ into $W^{1,p}_{\text{loc}}(\mathcal{N})$ (see [6]). That procedure applies to the more general situation in presence of the unit vector field $\boldsymbol{L}$ which is transversal to $\mathcal{N}$. Thus, straightening $\boldsymbol{L}$ in a neighbourhood of an arbitrary point of $\mathcal{N}$ such that $\partial/\partial \boldsymbol{L} \equiv \partial/\partial x_n$, $\mathcal{N}$ could be represented locally as a graph of a function $\Phi \in C^{1,1}$, after that (7) applies. We will refer in the sequel to that procedure as *taking trace on* $\mathcal{N}$ *along the* $\boldsymbol{L}$*-trajectories through the points of* $\mathcal{N}$.

These observations explain the assumption $\mu \in W^{2-1/p,p}(\mathcal{E})$ in Theorem 1. In fact, suppose $u \in W^{2,p}(\Omega)$ is a solution of ($\mathcal{MP}$). Then $u|_{\partial\Omega} \in W^{2-1/p,p}(\partial\Omega)$ and taking once again trace on the $(n-2)$-dimensional submanifold $\mathcal{E}$ of $\partial\Omega$ would give $(u|_{\partial\Omega})|_{\mathcal{E}} \in W^{2-2/p,p}(\mathcal{E})$. However, as it follows from (5) and (6), the higher-order regularity assumptions on the data near $\mathcal{E}$ ensure $\partial u/\partial \boldsymbol{\ell} \in W^{2-1/p,p}(\Sigma \cap \partial\Omega)$ and since $\boldsymbol{\ell}$ is strictly transversal to $\mathcal{E}$ by (3), we have really $u|_{\mathcal{E}} \in W^{2-1/p,p}(\mathcal{E})$.

3 Proof of the main result

The statement of Theorem 1 will follow by the corresponding results *away* (Lemma 3) and *near* (Lemma 4) the set of tangency $\mathcal{E}$. Fix hereafter $\Sigma' \subset \Sigma'' \subset \Sigma$ to be *closed* neighbourhoods of $\mathcal{E}$ in $\overline{\Omega}$.

Lemma 3 *Suppose* (1), (2) *and let* $u \in W^{2,q}(\Omega)$ *be a strong solution to* ($\mathcal{MP}$) *with* $f \in L^p(\Omega)$ *and* $\varphi \in W^{1-1/p,p}(\partial\Omega)$ *where* $p \in [q,\infty)$.

Then $u \in W^{2,p}(\Omega \setminus \Sigma')$ *and there is an absolute constant C such that*

$$\|u\|_{W^{2,p}(\Omega\setminus\Sigma')} \leq C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\varphi\|_{W^{1-1/p,p}(\partial\Omega)}\right).$$

Proof The problem ($\mathcal{MP}$) is a *regular* oblique derivative problem out of Σ', with a field $\boldsymbol{\ell}$ *strictly* transversal to $\partial\Omega$ and pointing into Ω on $\partial\Omega^-\backslash\Sigma'$ and out of Ω on $\partial\Omega^+\backslash\Sigma'$. The claims follow from or Theorem 2.3.1 of [7]. □

Lemma 4 *Assume* (1)–(3) *and let* $u \in W^{2,q}(\Omega)$ *be a strong solution to* ($\mathcal{MP}$) *with* $f \in \mathcal{F}^p(\Omega,\Sigma)$, $\varphi \in \Phi^p(\partial\Omega,\Sigma)$ *and* $\mu \in W^{2-1/p,p}(\mathcal{E})$ *where* $p \in [q,\infty)$.

Then $u \in W^{2,p}(\Sigma'')$ *and*

$$\|u\|_{W^{2,p}(\Sigma'')} \leq C\left(\|u\|_{L^p(\Omega)} + \|f\|_{\mathcal{F}^p(\Omega,\Sigma)} + \|\varphi\|_{\Phi^p(\partial\Omega,\Sigma)} + \|\mu\|_{W^{2-1/p,p}(\mathcal{E})}\right). \quad (8)$$

Proof Turning back to the neighbourhood Γ of $\partial\Omega$ and the extension of $\boldsymbol{\ell}$, we recall $\boldsymbol{\tau}|_{\mathcal{N}} \equiv \boldsymbol{L}|_{\mathcal{N}}$ and $\left|\boldsymbol{\tau}|_{\mathcal{N}}\right| = 1$. Therefore, there exists a closed neighbourhood U of $\mathcal{N}$,

$$U := \left\{x \in \Sigma \cap \Gamma: \quad |\boldsymbol{\tau}(x)| \geq 1/2\right\}$$

and setting $\boldsymbol{\tau}'(x) := \boldsymbol{\tau}(x)/|\boldsymbol{\tau}(x)| \forall x \in U$, we get the *unit* vector field $\boldsymbol{\tau}'$ coinciding with $\boldsymbol{\tau}$ on $\mathcal{N}$. The strict transversality of $\boldsymbol{\tau}'$ to $\mathcal{N}$ assures that any point $\overline{x} \in U$ can be reached from a unique $\overline{x}' \in \mathcal{N}$ along a trajectory of $\boldsymbol{\tau}'$ in the positive/negative direction. Setting $t \to \boldsymbol{\psi}_{\boldsymbol{\tau}'}(t,\overline{x})$ for the integral curve of $\boldsymbol{\tau}'$ through $\overline{x}$, we have $\overline{x} = \boldsymbol{\psi}_{\boldsymbol{\tau}'}(\xi,\overline{x}')$, $\overline{x}' \in \mathcal{N}$, $\xi \in \mathbb{R}$ and $\operatorname{sign}(\xi) = \operatorname{sign}(\gamma(y(\overline{x})))$ (see Fig. 3b). Introduce new coordinates $(\xi,\eta,\zeta) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-2}$ in U as follows. For any $\overline{x} \in U$ we set $\xi(\overline{x}) \in \mathbb{R}$ to be the length (with sign) of the $\boldsymbol{\tau}'$-trajectory connecting $\overline{x}$ with the unique $\overline{x}' \in \mathcal{N}$, i.e., $\overline{x} = \boldsymbol{\psi}_{\boldsymbol{\tau}'}(\xi(\overline{x}),\overline{x}')$ and $\operatorname{sign}(\xi(\overline{x})) = \operatorname{sign}(\gamma(y(\overline{x})))$. Define further $\eta(\overline{x}) := \operatorname{dist}(\overline{x}',\partial\Omega) = \operatorname{dist}(\boldsymbol{\psi}_{\boldsymbol{\tau}'}(-\xi(\overline{x}),\overline{x}),\partial\Omega)$. Finally, $\zeta(\overline{x}) \in \mathcal{E}$ is given by $\zeta(\overline{x}) := y\left(\boldsymbol{\psi}_{\boldsymbol{\tau}'}(-\xi(\overline{x}),\overline{x})\right) \in \mathcal{E}$.

Let $\mathcal{S}, \partial\mathcal{S} \in C^\infty$, be the convex domain in the (η,ξ)-plane as given on Fig. 3a. Set $\Omega_\delta := \left\{x \in U: \zeta(x) \in \mathcal{E}, ((\eta(x),\xi(x)) \in \delta\cdot\mathcal{S}\right\}$ for $\delta \in (0,\delta_0]$ with $\delta_0 \ll 1$ and $\delta\cdot\mathcal{S}$ standing for the dilation of $\mathcal{S}$ of factor δ. Indeed, $\overline{\Omega_\delta} \subset U$, $\partial\Omega_\delta \in C^{1,1}$ and if δ_0 is small enough then the field $\boldsymbol{L}$ is tangential to $\partial\Omega_\delta$ *only* at the points of $\mathcal{E}$ and these of $\mathcal{E}_\delta := (\mathcal{N}\cap\partial\Omega_\delta)\backslash\mathcal{E} = \mathcal{N}\cap\partial\Omega_\delta\cap\Omega$ and points outwards (inwards) Ω_δ at $x \in \partial\Omega_\delta\backslash(\mathcal{E}\cup\mathcal{E}_\delta)$ when $y(x) \in \partial\Omega^+$ ($y(x) \in \partial\Omega^-$). We define further

$$\mathcal{N}_\delta := \mathcal{N}\cap\Omega_\delta, \quad \partial\mathcal{N}_\delta := \mathcal{E}\cup\mathcal{E}_\delta.$$

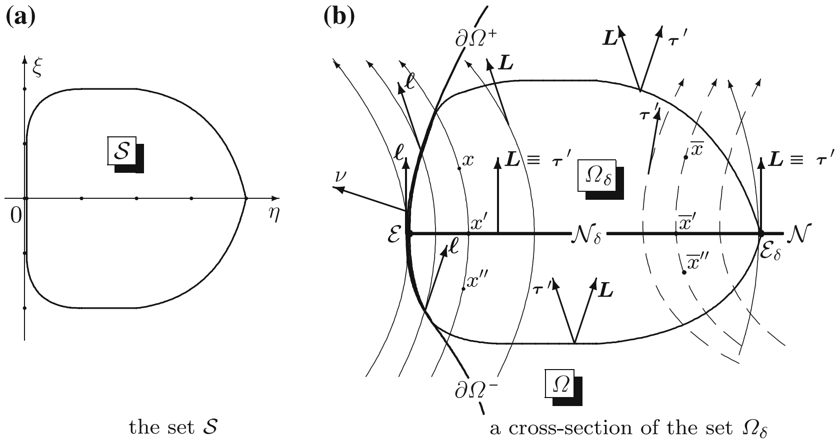

Fig. 3 The *dashed curves* represent trajectories of the field $\boldsymbol{\tau}'$, parameterized by $t \to \boldsymbol{\psi}_{\boldsymbol{\tau}'}(t, \overline{x}), \overline{x}' \in \mathcal{N}, \overline{x} = \boldsymbol{\psi}_{\boldsymbol{\tau}'}(\xi(\overline{x}), \overline{x}'), \overline{x}'' = \boldsymbol{\psi}_{\boldsymbol{\tau}'}(\xi(\overline{x}''), \overline{x}')$. The other curves are $\boldsymbol{L}$-trajectories parameterized by $t \to \boldsymbol{\psi}_{\boldsymbol{L}}(t, x)$ and $x = \boldsymbol{\psi}_{\boldsymbol{L}}(s(x'), x'), x'' = \boldsymbol{\psi}_{\boldsymbol{L}}(s(x''), x')$ with $x' \in \mathcal{N}$.

Each point $x \in U$ can be reached from $x' \in \mathcal{N}$ through an $\boldsymbol{L}$-trajectory (see Fig. 3b). Setting $t \to \boldsymbol{\psi}_{\boldsymbol{L}}(t, x)$ for its parameterization, for each $x \in U$ there exists a unique value $s(x) \in C^{1,1}(U)$ of the parameter such that $\boldsymbol{\psi}_{\boldsymbol{L}}(-s(x), x) = x' \in \mathcal{N}$ and without loss of generality we may assume $|s(x)| \le \delta \forall x \in \Omega_\delta$. Now, for any $x' \in \mathcal{N}$ define the trace of $f \in \mathcal{F}^p(\Omega.\Sigma)$ on $\mathcal{N}$ along the $\boldsymbol{L}$-trajectories by

$$\widetilde{f}(x') := f(x) - \int_0^{s(x)} \frac{\partial f}{\partial \boldsymbol{L}} \circ \boldsymbol{\psi}_{\boldsymbol{L}}(t, x')\mathrm{d}t, \quad x \in U.$$

It follows from Remark 2 that $\widetilde{f}$ is well-defined on $\mathcal{N}$ and $\widetilde{f} \in L^p(\mathcal{N})$. In the same manner, $u \in W^{2,q}(\Omega)$ and the trace $\widetilde{u}(x') = u(x)|_{\mathcal{N}} := u \circ \boldsymbol{\psi}_{\boldsymbol{L}}(-s(x), x)$ does exist.

Setting

$$v(x) := \partial u(x)/\partial \boldsymbol{L} \quad \forall\, x \in \Omega_\delta$$

it is obvious that

$$\begin{aligned} u(x) = {} & \widetilde{u}(x') + \int_0^{s(x)} v \circ \boldsymbol{\psi}_{\boldsymbol{L}}(t, x')\mathrm{d}t \\ & = u \circ \boldsymbol{\psi}_{\boldsymbol{L}}(-s(x), x) + \int_0^{s(x)} v \circ \boldsymbol{\psi}_{\boldsymbol{L}}(t - s(x), x)\mathrm{d}t, \quad \forall x \in \Omega_\delta. \end{aligned} \tag{9}$$

To get the improving-of-summability property for $u(x)$ we will derive it for $\widetilde{u}(x')$ and $v(x)$, and we suppose $p > q$. Consider the action of $\mathcal{L}$ on the functions defined in U which are constant on almost every $\boldsymbol{L}$-trajectory through $\mathcal{N}$. This defines a second order operator $\mathcal{L}'$ on the $C^{1,1}$-smooth manifold $\mathcal{N}$, which is uniformly elliptic by virtue of (1) and the strict transversality of $\boldsymbol{L}$ to $\mathcal{N}$. This way, $\widetilde{u}(x')$ is a $W^{2,q}(\mathcal{N})$-solution of the following Dirichlet problem on the manifold $\mathcal{N}_\delta$

$$\begin{cases} \mathcal{L}'\widetilde{u} = \widetilde{F}' & \text{a.e. } \mathcal{N}_\delta, \end{cases} \qquad \widetilde{u}|_{\partial\mathcal{N}_\delta} = \begin{cases} \mu \text{ on } \mathcal{E}, \\ u \text{ on } \mathcal{E}_\delta. \end{cases} \tag{10}$$

To get a local representation for the operator $\mathcal{L}'$ we suppose, without loss of generality, that the field $\boldsymbol{L}$ is locally straighten in a neighbourhood of a point $x_0 \in \mathcal{N}$ such that $\partial/\partial\boldsymbol{L} \equiv \partial/\partial x_n$ and $\mathcal{N}$ has the form $\{x_n = 0\}$ near x_0. Thus, setting $x' = (x_1, \ldots, x_{n-1}) \in \mathcal{O}' \subset \mathcal{N}$, we have $v = \partial u/\partial\boldsymbol{L} = \partial u/\partial x_n$ and

$$\begin{aligned}\mathcal{L}'\widetilde{u}(x') &\equiv \sum_{i,j=1}^{n-1} a^{ij}(x',0)D_{x'_ix'_j}\widetilde{u}(x') + \sum_{i=1}^{n-1} b^i(x',0)D_{x'_i}\widetilde{u}(x') + c(x',0)\widetilde{u}(x') \\ &= \widetilde{F}'(x') := \widetilde{f}(x') - \sum_{i=1}^{n-1} a^{in}(x',0)(\widetilde{D_{x'_i}v})(x') - a^{nn}(x',0)(\widetilde{D_{x_n}v})(x') \\ &\quad - b^n(x',0)\widetilde{v}(x'),\end{aligned} \tag{11}$$

where the "tilde" over a function means its trace value on $\mathcal{N}$ taken along the $\boldsymbol{L}$-trajectories in the sense of (7). We have $f \in \mathcal{F}^p(\Omega, \Sigma)$ and therefore $\widetilde{f} \in L^p(\mathcal{N})$ as it follows from Remark 2 (2). Further, $v \in W^{2,q}(\Sigma)$ in view of Remark 2 (1) and thus $\widetilde{v}$, $\widetilde{D_x v} \in L^r(\mathcal{N})$ with $r = (n-1)q/(n-q)$ if $q < n$ and arbitrary $r > 1$ when $q \geq n$ (cf. Theorems 6.4.1 and 6.4.2 of [6]). This means $\widetilde{F}' \in L^{q'}(\mathcal{N}_\delta)$ with

$$q' = \begin{cases} \min\left\{p, \frac{(n-1)q}{n-q}\right\}, & \text{if } q < n, \\ p, & \text{otherwise.} \end{cases} \tag{12}$$

Further on, $\mu \in W^{2-1/p,p}(\mathcal{E})$ and $u|_{\mathcal{E}_\delta} \in W^{2-1/p,p}$ by Lemma 3. and the L^p-theory (see [2]) yields that the solution $\widetilde{u}$ of (10) belongs to $W^{2,q'}(\mathcal{N}_\delta)$ with $q' > q$.

To get increasing of summability for $v = \partial u/\partial\boldsymbol{L}$ also, we recall (see (5)) that the function v is a $W^{2,q}$-solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}v = \partial f/\partial\boldsymbol{L} + 2a^{ij}D_jL^kD_{ki}u + (a^{ij}D_{ij}L^k + b^iD_iL^k)D_ku \\ \qquad -(\partial a^{ij}/\partial\boldsymbol{L})D_{ij}u - (\partial b^i/\partial\boldsymbol{L})D_iu - (\partial c/\partial\boldsymbol{L})u \qquad \text{a.e. } \Omega_\delta, \\ v = \varphi \quad \text{on } \partial\Omega_\delta \cap \partial\Omega, \qquad v = \partial u/\partial\boldsymbol{L} \quad \text{on } \partial\Omega_\delta \cap \Omega. \end{cases} \tag{13}$$

We have $\partial u/\partial\boldsymbol{L} \in W^{2-1/p,p}(\partial\Omega_\delta \cap \Omega)$ by Lemma 3 and Remark 2 (1), while $\varphi \in W^{2-1/p,p}$ $(\partial\Omega_\delta \cap \partial\Omega)$. Take the second derivatives of u from (9) and substitute them into the right-hand side of the equation above. This rewrites (13) into

$$\begin{cases} \mathcal{L}v = F(x) + \displaystyle\int_0^{s(x)} (\mathcal{L}_2 v) \circ \boldsymbol{\psi}_{\boldsymbol{L}}(t, x')\mathrm{d}t, \qquad \text{a.e. } \Omega_\delta, \\ v = \varphi \in W^{2-1/p,p}, \quad \text{on } \partial\Omega_\delta \cap \partial\Omega, \qquad v \in W^{2-1/p,p} \quad \text{on } \partial\Omega_\delta \cap \Omega \end{cases} \tag{14}$$

with

$$F(x) := \partial f(x)/\partial\boldsymbol{L} + \mathcal{L}_1 v(x) + \widetilde{\mathcal{L}}_2\widetilde{u}(x').$$

Here $\mathcal{L}_i, i = 1, 2,$ is a differential operator of order i with L^∞-coefficients and $\widetilde{L}_2$ is a second-order differential operator over the manifold $\mathcal{N}_\delta$. We have $\widetilde{u} \in W^{2,q'}(\mathcal{N}_\delta)$ whence $\widetilde{L}_2\widetilde{u} \in L^{q'}(\Omega_\delta)$. Moreover, $v \in W^{2,q}(\Omega_\delta)$ and Sobolev's imbedding theorem implies $\mathcal{L}_1 v \in L^r(\Omega_\delta)$ with $r = nq/(n-q)$ when $q < n$ and any $r > 1$ otherwise. Since $\partial f/\partial\boldsymbol{L} \in L^p(\Sigma)$

by hypotheses, we get $F \in L^{q''}(\Omega_\delta)$ with $q'' = \min\{p, r, q'\}$. It is clear that $q'' = q'$ with q' given by (12), $q' > q$ and therefore $F \in L^{q'}(\Omega_\delta)$.

We will prove now that the solution $v \in W^{2,q}(\Omega_\delta)$ of the non-local Dirichlet problem (14) with $F \in L^{q'}(\Omega_\delta)$, belongs to $W^{2,q'}(\Omega_\delta)$ when δ is chosen small enough. For, take any $r \in [q, q']$ and set $W^{2,r}_*(\Omega_\delta)$ for the Sobolev space $W^{2,r}(\Omega_\delta)$ equipped with the non-dimensional norm

$$\|u\|_{W^{2,r}_*(\Omega_\delta)} := \|u\|_{L^r(\Omega_\delta)} + \delta\|Du\|_{L^r(\Omega_\delta)} + \delta^2\|D^2u\|_{L^r(\Omega_\delta)}.$$

For an arbitrary $w \in W^{2,r}_*(\Omega_\delta)$ we have $\int_0^{s(x)}(\mathcal{L}_2 w)\circ \boldsymbol{\psi}_{\boldsymbol{L}}(t,x')dt \in L^r(\Omega_\delta)$ and therefore there exists a unique solution $\mathcal{F}w \in W^{2,r}_*(\Omega_\delta)$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}(\mathcal{F}w) = F(x) + \displaystyle\int_0^{s(x)} (\mathcal{L}_2 w)\circ \boldsymbol{\psi}_{\boldsymbol{L}}(t,x')\mathrm{d}t \qquad \text{a.a. } x \in \Omega_\delta, \\ \mathcal{F}w = \varphi \in W^{2-1/p,p} \quad \text{on } \partial\Omega_\delta \cap \partial\Omega, \quad \mathcal{F}w = \partial u/\partial \boldsymbol{L} \in W^{2-1/p,p} \quad \text{on } \partial\Omega_\delta \cap \Omega. \end{cases}$$

This defines a map $\mathcal{F}\colon\ W^{2,r}_*(\Omega_\delta) \to W^{2,r}_*(\Omega_\delta)$ which turns out to be a *contraction* if $\delta > 0$ is taken small enough. In fact, for any $w_1,\ w_2 \in W^{2,r}_*(\Omega_\delta)$ we have

$$\begin{cases} \mathcal{L}(\mathcal{F}w_1 - \mathcal{F}w_2) = \displaystyle\int_0^{s(x)} (\mathcal{L}_2(w_1 - w_2))\circ \boldsymbol{\psi}_{\boldsymbol{L}}(t,x')\mathrm{d}t \qquad \text{a.a. } x \in \Omega_\delta, \\ \mathcal{F}w_1 - \mathcal{F}w_2 = 0 \quad \text{on } \partial\Omega_\delta. \end{cases} \tag{15}$$

In order to employ the L^r-a priori estimates for (15) (cf. [2]) we have to control the dependence on δ therein. That is why, we first dilate Ω_δ into $\delta^{-1}\Omega_\delta$ for which $\partial(\delta^{-1}\Omega_\delta) \in C^{1,1}$ *uniformly* in δ, and then apply the cited estimates. A procedure, similar to the one from the Proof of Lemma 2.2 and Equation (2.12) in [12] gives

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\|_{W^{2,r}_*(\Omega_\delta)} \le C\delta^2 \left\| \int_0^{s(x)} (\mathcal{L}_2(w_1 - w_2))\circ \boldsymbol{\psi}_{\boldsymbol{L}}(t,x')\mathrm{d}t \right\|_{L^r(\Omega_\delta)} \tag{16}$$

with a constant C independent of $\delta > 0$. Moreover, $\int_0^{s(x)} g\circ \boldsymbol{\psi}_{\boldsymbol{L}}(t,x')\mathrm{d}t \in L^r(\Omega_\delta)$ for each $g(x) \in L^r(\Omega_\delta)$ and application of Jensen's integral inequality leads to

$$\left\| \int_0^{s(x)} g\circ \boldsymbol{\psi}_{\boldsymbol{L}}(t,x')\mathrm{d}t \right\|_{L^r(\Omega_\delta)} \le C \max_{\Omega_\delta} |s(x)|\|g\|_{L^r(\Omega_\delta)} \le C\delta\|g\|_{L^r(\Omega_\delta)}. \tag{17}$$

This way, remembering $|s(x)| \le \delta \forall x \in \Omega_\delta$, (16) rewrites as

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\|_{W^{2,r}_*(\Omega_\delta)} \le C\delta^3\,\|\mathcal{L}_2(w_1 - w_2)\|_{L^r(\Omega_\delta)} \le C\delta\|w_1 - w_2\|_{W^{2,r}_*(\Omega_\delta)},$$

whence

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\|_{W^{2,r}_*(\Omega_\delta)} \le K\|w_1 - w_2\|_{W^{2,r}_*(\Omega_\delta)}, \quad K < 1$$

if $\delta > 0$ is fixed small enough. Therefore, $\mathcal{F}$ is a contraction mapping from $W^{2,r}_*(\Omega_\delta)$ into itself for each $r \in [q, q']$ if $\delta > 0$ is chosen sufficiently small. The unique fixed point of $\mathcal{F}$ belongs to $W^{2,r}(\Omega_\delta)$ for each $r \in [q, q']$, and since $v \in W^{2,q}(\Omega_\delta)$ solves (14) and is therefore already a fixed point of $\mathcal{F}$, we conclude $v \in W^{2,q'}(\Omega_\delta)$.

Indeed, this yields $u \in W^{2,q'}(\Omega_\delta)$ with $q' > q$ on the base of $\widetilde{u} \in W^{2,q'}(\mathcal{N}_\delta)$ and (9). To arrive at $u \in W^{2,p}(\Omega_\delta)$ it suffices to repeat the above procedure finitely many times with

q' instead of q until q' becomes equal to p. Noting that Lemma 3 remains valid with Σ' replaced by Ω_δ, we get $u \in W^{2,p}(\Sigma'')$ as Lemma 4 claims.

To derive the bound (8), we note that (9), (17) and $|s(x)| \le \delta\ \forall x \in \Omega_\delta$ imply

$$\|D^2u\|_{L^p(\Omega_\delta)} \le \|\widetilde{\mathcal{L}}'_2\widetilde{u}\|_{L^p(\Omega_\delta)} + C\|v\|_{W^{1,p}(\Omega_\delta)} + C\delta\|D^2v\|_{L^p(\Omega_\delta)}, \tag{18}$$

where C is independent of δ and $\widetilde{\mathcal{L}}'_2$ is a second-order differential operator over the manifold $\mathcal{N}_\delta$ acting on $\widetilde{u} \in W^{2,p}(\mathcal{N}_\delta)$.

Set $M := \|f\|_{\mathcal{F}^p(\Omega,\Sigma)} + \|\varphi\|_{\Phi^p(\partial\Omega,\Sigma)} + \|\mu\|_{W^{2-1/p,p}(\mathcal{E})}$ for the sake of simplicity. Passing to $\delta^{-1}\Omega_\delta$ and using that v solves the problem (13), a procedure similar to that already employed above gives

$$\|D^2v\|_{L^p(\Omega_\delta)} \le C'(\delta)\left(M + \|\partial u/\partial \boldsymbol{L}\|_{W^{2-1/p,p}(\partial\Omega_\delta\cap\Omega)}\right) + C\|u\|_{W^{2,p}(\Omega_\delta)},$$

while

$$\|\partial u/\partial \boldsymbol{L}\|_{W^{2-1/p,p}(\partial\Omega_\delta\cap\Omega)} \le C\|\partial u/\partial \boldsymbol{L}\|_{W^{2,p}(\Sigma\setminus\Omega_\delta)} \le C'(\delta)\left(M + \|u\|_{L^p(\Omega)}\right)$$

by (6), whence

$$\|D^2v\|_{L^p(\Omega_\delta)} \le C'(\delta)\left(M + \|u\|_{W^{1,p}(\Omega_\delta)}\right) + C\|D^2u\|_{L^p(\Omega_\delta)}. \tag{19}$$

Further on, extending $\widetilde{\mathcal{L}}'_2\widetilde{u}$ as constant in Ω_δ along the $\boldsymbol{L}$-trajectories through the points of $\mathcal{N}_\delta$, and using $|s(x)| \le \delta$ for each $x \in \Omega_\delta$, we get

$$\begin{aligned}\|\widetilde{\mathcal{L}}'_2\widetilde{u}\|_{L^p(\Omega_\delta)} &\le C\delta^{1/p}\|\widetilde{\mathcal{L}}'_2\widetilde{u}\|_{L^p(\mathcal{N}_\delta)} \le C\delta^{1/p}\|\widetilde{u}\|_{W^{2,p}(\mathcal{N}_\delta)}\\ &\le C'(\delta)\left(M + \|u\|_{W^{2-1/p,p}(\mathcal{E}_\delta)}\right) + C\delta^{1/p}\|\widetilde{F}'\|_{L^p(\mathcal{N}_\delta)}\\ &\le C'(\delta)\left(M + \|u\|_{L^p(\Omega)}\right) + C\delta^{1/p}\|\widetilde{F}'\|_{L^p(\mathcal{N}_\delta)}\end{aligned} \tag{20}$$

as consequence of the L^p-estimates for the problem (10) and Lemma 3.

Turning to the local coordinate system centered at $x_0 \in \mathcal{N}_\delta$ (see (10) and (11)) in which $\partial/\partial\boldsymbol{L} \equiv \partial/\partial x_n$, we define the function

$$F'(x', x_n) := f(x', x_n) - \sum_{i=1}^{n} a^{in}(x', x_n)D_i v(x', x_n) - b^n(x', x_n)v(x', x_n).$$

It is clear that the trace of $F'(x', x_n)$ on $\mathcal{N}_\delta$ along the $\boldsymbol{L}$-trajectories is exactly $\widetilde{F}'$ given by (11) and [12, Equation (2.9)] gives

$$\delta^{1/p}\|\widetilde{F}'\|_{L^p(\mathcal{N}_\delta)} \le C\left(\|F'\|_{L^p(\Omega_\delta)} + \delta\|\partial F'/\partial\boldsymbol{L}\|_{L^p(\Omega_\delta)}\right).$$

This way (20) becomes

$$\begin{aligned}\|\widetilde{\mathcal{L}}'_2\widetilde{u}\|_{L^p(\Omega_\delta)} &\le C'(\delta)\left(M + \|u\|_{W^{1,p}(\Omega_\delta)} + \|v\|_{W^{1,p}(\Omega_\delta)}\right) + C\delta\|D^2v\|_{L^p(\Omega_\delta)}\\ &\le C'(\delta)\left(M + \|u\|_{W^{1,p}(\Omega_\delta)} + \|v\|_{W^{1,p}(\Omega_\delta)}\right) + C\delta\|D^2u\|_{L^p(\Omega_\delta)}.\end{aligned} \tag{21}$$

It follows from (19) and (21) that (18) takes on the form

$$\|D^2u\|_{L^p(\Omega_\delta)} \le C'(\delta)\left(M + \|u\|_{W^{1,p}(\Omega_\delta)} + \|v\|_{W^{1,p}(\Omega_\delta)}\right) + C\delta\|D^2u\|_{L^p(\Omega_\delta)}$$

with C independent of δ. Fixing $\delta > 0$ small enough, we get into

$$\|u\|_{W^{2,p}(\Omega_\delta)} \le C\left(M + \|u\|_{W^{1,p}(\Omega_\delta)} + \|v\|_{W^{1,p}(\Omega_\delta)}\right). \tag{22}$$

The estimate (8) follows from (22) by interpolation. In fact, since δ is small we may suppose $\Omega_\delta \subset \Sigma' \subset \Sigma''$ and

$$\begin{aligned}\|u\|_{W^{2,p}(\Sigma'')} &\le \|u\|_{W^{2,p}(\Omega_\delta)} + \|u\|_{W^{2,p}(\Omega\setminus\Omega_\delta)}\\ &\le C\left(M + \|u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega_\delta)} + \|v\|_{W^{1,p}(\Omega_\delta)}\right)\\ &\le C\left(M + \|u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Sigma'')} + \|v\|_{W^{1,p}(\Sigma')}\right) \qquad (23)\end{aligned}$$

by virtue of (22) and Lemma 3 applied to the term $\|u\|_{W^{2,p}(\Omega\setminus\Omega_\delta)}$ with Ω_δ instead of Σ'. On the other hand, assuming some minimal smoothness of $\partial\Sigma'$ and $\partial\Sigma''$, the interpolation inequality implies

$$\|v\|_{W^{1,p}(\Sigma')} \le \varepsilon\|v\|_{W^{2,p}(\Sigma')} + C(\varepsilon)\|v\|_{L^p(\Sigma')}, \quad \forall\varepsilon > 0,$$

while

$$\|v\|_{W^{2,p}(\Sigma')} \le C\left(M + \|u\|_{L^p(\Omega)} + \|u\|_{W^{2,p}(\Sigma'')}\right)$$

in view of (6). This way, (23) becomes

$$\|u\|_{W^{2,p}(\Sigma'')} \le \varepsilon\|u\|_{W^{2,p}(\Sigma'')} + C(\varepsilon)\left(M + \|u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Sigma'')}\right),$$

which reads

$$\|u\|_{W^{2,p}(\Sigma'')} \le C\left(M + \|u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Sigma'')}\right)$$

after choosing ε small enough. To get the estimate (8), it remains to apply once again the interpolation inequality to the term $\|u\|_{W^{1,p}(\Sigma'')}$. This completes the proof of Lemma 4. □

4 Concluding remarks

We will briefly sketch here some important consequences of the improving-of-integrability property and the a priori estimate (8) as stated in Theorem 1. The interested reader is referred to [12] for the proofs, while [14] provides for generalizations to the case of tangency set $\mathcal{E}$ which is no anymore a codimension one submanifold of $\partial\Omega$, but may have positive surface measure.

Maximum principle and uniqueness in $W^{2,p}(\Omega)$ for each $p > 1$.

Lemma 5 *Assume* (1)–(3), $c(x) \le 0$ *a.e.* Ω *and let* $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^1(\overline{\Omega})$ *satisfy*

$$\begin{cases} a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u \ge 0 & a.e.\ \Omega,\\ \partial u/\partial\boldsymbol{\ell} \le 0 \quad on\ \partial\Omega^+, \quad \partial u/\partial\boldsymbol{\ell} \ge 0 & on\ \partial\Omega^-, \quad u \le 0 \quad on\ \mathcal{E}.\end{cases}$$

Then $u(x) \le 0$ *on* $\overline{\Omega}$.

The unicity of the $W^{2,p}(\Omega)$-solutions to $(\mathcal{MP})$ for each $p > 1$ is a direct consequence of the maximum principle and the improving-of-summability property.

Corollary 6 *Assume* (1)–(3) *and* $c(x) \le 0$ *a.e.* Ω. *Let* $u,\ v \in W^{2,p}(\Omega)$ *be two solutions to* $(\mathcal{MP})$ *with* $p > 1$. *Then* $u \equiv v$ *in* $\overline{\Omega}$.

Refined A Priori Estimate and Unique Solvability in $W^{2,p}(\Omega)$ for each $p > 1$ when $c(x) \le 0$ a.e. Ω. In case the coefficient $c(x)$ of $\mathcal{L}$ is non-positive, the bound (8) could be considerably refined by dropping out $\|u\|_{L^p(\Omega)}$ from the right-hand side.

Lemma 7 *Assume* (1)–(3) *and* $c(x) \leq 0$ *a.e.* Ω*. Let* $u \in W^{2,p}(\Omega)$, $p > 1$, *be a strong solution to* $(\mathcal{MP})$ *with* $f \in \mathcal{F}^p(\Omega, \Sigma)$, $\varphi \in \Phi^p(\partial\Omega, \Sigma)$ *and* $\mu \in W^{2-1/p,p}(\mathcal{E})$.

Then there exists a constant C*, depending on known quantities only, such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|f\|_{\mathcal{F}^p(\Omega,\Sigma)} + \|\varphi\|_{\Phi^p(\partial\Omega,\Sigma)} + \|\mu\|_{W^{2-1/p,p}(\mathcal{E})}\right). \tag{24}$$

The a priori estimate (24) yields strong solvability of the Poincaré problem $(\mathcal{MP})$ in $W^{2,p}(\Omega)$ for arbitrary $p > 1$ whenever the uniqueness hypotheses of Corollary 6 hold. In fact, approximating $(\mathcal{MP})$ by problems with C^∞-smooth data and using the existence results from [3,4] or [22], (24)[1] gives

Theorem 8 *Assume* (1)–(3) *and* $c(x) \leq 0$ *a.e.* Ω*. Then, for each* $p > 1$ *the Poincaré problem* $(\mathcal{MP})$ *is uniquely solvable in* $W^{2,p}(\Omega)$ *for arbitrary* $f \in \mathcal{F}^p(\Omega, \Sigma)$, $\varphi \in \Phi^p(\partial\Omega, \Sigma)$ *and* $\mu \in W^{2-1/p,p}(\mathcal{E})$.

$(\mathcal{MP})$ is a problem of Fredholm type with index zero. Let $p > 1$ be arbitrary and set $\mathcal{W}^{2,p}(\Omega, \Sigma)$ for the Banach space of functions $u \in W^{2,p}(\Omega)$ such that $\partial u/\partial \boldsymbol{L} \in W^{2,p}(\Sigma)$ and normed by $\|u\|_{\mathcal{W}^{2,p}(\Omega,\Sigma)} := \|u\|_{W^{2,p}(\Omega)} + \|\partial u/\partial \boldsymbol{L}\|_{W^{2,p}(\Sigma)}$. Define the *kernel* and the *range* of $(\mathcal{MP})$ by

$$\mathcal{K}_p := \left\{u \in \mathcal{W}^{2,p}(\Omega, \Sigma) : \quad \mathcal{L}u = 0 \text{ a.e. } \Omega, \quad \partial u/\partial \boldsymbol{\ell} = 0 \text{ on } \partial\Omega, \quad u = 0 \text{ on } \mathcal{E}\right\},$$
$$\mathcal{R}_p := \mathcal{F}^p(\Omega, \Sigma) \times \Phi^p(\partial\Omega, \Sigma) \times W^{2-1/p,p}(\mathcal{E}).$$

Theorem 9 *Under the hypotheses* (1)–(3)*, for any* $p \in (1, \infty)$ *there exists a closed subspace* $\widetilde{\mathcal{R}}_p$ *of finite codimension in* $\mathcal{R}_p$ *such that for arbitrary* $(f, \varphi, \mu) \in \widetilde{\mathcal{R}}_p$ *the modified Poincaré problem* $(\mathcal{MP})$ *has a solution* $u \in \mathcal{W}^{2,p}(\Omega)$*. Moreover,* $\dim \mathcal{K}_p = \operatorname{codim}_{\mathcal{R}_p} \widetilde{\mathcal{R}}_p$ *and if, in particular,* $c(x) \leq 0$ *a.e.* Ω*, then* $\mathcal{K}_p = \{0\}$, $\widetilde{\mathcal{R}}_p \equiv \mathcal{R}_p$ *and* $(\mathcal{MP})$ *is uniquely solvable for arbitrary* $(f, \varphi, \mu) \in \mathcal{R}_p$.

In terms of the Poincaré problem $(\mathcal{MP})$, Theorem 9 sounds like

Corollary 10 *Suppose* (1)–(3) *and let* $p > 1$ *be any number. Then, either*

(A) *the homogeneous problem*

$$\mathcal{L}u = 0 \quad a.e.\ \Omega, \qquad \mathcal{B}u = 0 \quad on\ \partial\Omega, \quad u = 0 \quad on\ \mathcal{E}$$

has only the trivial solution and then the non-homogeneous problem $(\mathcal{MP})$ *is uniquely solvable in* $W^{2,p}(\Omega)$ *for arbitrary* $(f, \varphi, \mu) \in \mathcal{F}^p(\Omega, \Sigma) \times \Phi^p(\partial\Omega, \Sigma) \times W^{2-1/p,p}(\mathcal{E})$*; or*

(B) *the homogeneous problem admits non-trivial solutions which span a subspace of* $W^{2,p}(\Omega)$ *of finite dimension* $k > 0$*. Then the non-homogeneous problem* $(\mathcal{MP})$ *is solvable only for those* $(f, \varphi, \mu) \in \mathcal{F}^p(\Omega, \Sigma) \times \Phi^p(\partial\Omega, \Sigma) \times W^{2-1/p,p}(\mathcal{E})$ *which satisfy* k *complementary conditions.*

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[1] We refer the reader to [12] for a direct approach to the existence problem for $(\mathcal{MP})$.

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